

On intra-regular and left regular and left duo ordered Γ -semigroups

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Abstract. For an intra-regular or a left regular and left duo ordered Γ -semigroup M , we describe the principal filter of M which plays an essential role in the structure of this type of po - Γ -semigroups. We also prove that an ordered Γ -semigroup M is intra-regular if and only if the ideals of M are semiprime and it is left (right) regular and left (right) duo if and only if the left (right) ideals of M are semiprime.

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1 Introduction and prerequisites

Croisot, who used the term “inversive” instead of “regular”, connects the matter of decomposition of a semigroup with the regularity and semiprime conditions [1]. A semigroup S is said to be left (resp. right) regular if for every $a \in S$ there exists $x \in S$ such that $a = xa^2$ (resp. $a = a^2x$). That is, if $a \in Sa^2$ (resp. $a \in a^2S$) for every $a \in S$ which is equivalent to saying that $A \subseteq A^2S$ (resp. $A \subseteq SA^2$) for every $A \subseteq S$. A semigroup S is said to be intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$. In other words, if $a \in Sa^2S$ for every $a \in S$ or $A \subseteq SA^2S$ for every $A \subseteq S$. For decompositions of an intra-regular, left regular or both left regular and right regular semigroup we refer to [2, 10]. The concepts of intra-regular ordered semigroup and of right regular ordered semigroups have been introduced in [3, 4] in which the decomposition of an intra-regular ordered semigroup into simple components and the decomposition of a right regular and right duo ordered semigroup into right simple components has been studied. The principal filter of S has a very simple form, both for ordered and non-ordered case of semigroups, and it plays an essential role in the decompositions.

For two nonempty sets M and Γ , we denote by $A\Gamma B$ the set containing the elements of the form $a\gamma b$ where $a \in A$, $\gamma \in \Gamma$ and $b \in B$. That is, we define

$$A\Gamma B := \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$

Then M is called a Γ -semigroup if the following assertions are satisfied:

- (1) $M\Gamma M \subseteq M$;
- (2) $a\gamma(b\mu c) = (a\gamma b)\mu c$ for all $a, b, c \in M$ and all $\gamma, \mu \in \Gamma$;
- (3) if $a, b, c, d \in M$ and $\gamma, \mu \in \Gamma$ such that $a = c$, $\gamma = \mu$ and $b = d$, then $a\gamma b = c\mu d$.

An *ordered Γ -semigroup* (shortly, *po- Γ -semigroup*) is a Γ -semigroup M with an order relation “ \leq ” on M such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in M$. A nonempty subset A of M is called a *subsemigroup* of M if, for every $a, b \in A$ and every $\gamma \in \Gamma$, we have $a\gamma b \in A$. A subsemigroup F of M is called a *filter* of M if (1) for every $a, b \in F$ and every $\gamma \in \Gamma$ such that $a\gamma b \in F$, we have $a \in F$ and $b \in F$ and (2) if $a \in F$ and $M \ni b \geq a$, then $b \in F$. For an element x of M , we denote by $N(x)$ the *filter of M generated by x* (that is, the least with respect to the inclusion relation filter of M containing x). A nonempty subset A of M is called a *left* (resp. *right*) *ideal* of M if (1) $M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$) and (2) if $a \in A$ and $M \ni b \leq a$, then $b \in A$. It is called an *ideal* or (*two-sided ideal*) of M if it is both a left and right ideal of M . A *po- Γ -semigroup* M is called *left* (resp. *right*) *duo* if the left (resp. right) ideals of M are two-sided. A subset T of M is called *semiprime* if for every $x \in M$ and every $\gamma \in \Gamma$ such that $x\gamma x \in T$, we have $x \in T$. For a subset H of M we denote by $(H]$ the subset of M defined by $(H] = \{t \in M \mid t \leq a \text{ for some } t \in H\}$. We clearly have $M = (M]$, and for any subsets A, B, C, D of M , we have $A \subseteq (A] = ((A])$; if $A \subseteq B$, then $(A] \subseteq (B]$; if $A \subseteq B$ and $C \subseteq D$, then $(A\Gamma C] \subseteq (B\Gamma D]$; $(A]\Gamma(B] \subseteq (A\Gamma B]$; and $((A]\Gamma(B]) = ((A]\Gamma B] = (A\Gamma(B]) = (A\Gamma B]$. As we know, some results on semigroups (ordered semigroups) can be transferred into Γ -semigroups (*po- Γ -semigroups*) just putting a Gamma in the appropriate place, while for some other results the transfer is not easy. A Γ -semigroup M is called *intra-regular* if $a \in M\Gamma a\Gamma a\Gamma M$ for every $a \in M$, equivalently if $A \subseteq M\Gamma A\Gamma A\Gamma M$ for every $A \subseteq M$. It is called *left* (resp. *right*) *regular* if $a \in M\Gamma a\Gamma a$ (resp. $a \in \Gamma a\Gamma M$) for every $a \in M$ or $A \subseteq M\Gamma A\Gamma A$ (resp. $A \subseteq \Gamma A\Gamma M$) for every $A \subseteq M$. An ordered Γ -semigroup M is called *intra-regular* if for every $a \in M$ we have $a \in (M\Gamma a\Gamma a\Gamma M]$, equivalently if for every $A \subseteq M$ we have $A \subseteq (M\Gamma A\Gamma A\Gamma M]$. An ordered Γ -semigroup M is called *left* (resp. *right*)

regular if $a \in (M\Gamma a\Gamma a]$ (resp. $(a \in a\Gamma a\Gamma M]$) for every $a \in M$, equivalently if $A \subseteq (M\Gamma A\Gamma A]$ (resp. $A \subseteq (A\Gamma A\Gamma M]$) for every $A \subseteq M$. Although some interesting results on Γ -semigroups are obtained with these definitions, these definitions fail to describe the principal filter of intra-regular, left regular and right regular Γ -semigroups (ordered Γ -semigroups) which play an essential role in the investigation. To overcome this difficulty, in [8] a new definition of intra-regular and of left regular Γ -semigroups has been introduced. The intra-regular Γ -semigroup has been defined as a Γ -semigroup such that $a \in M\Gamma a\gamma a\Gamma M$ for each $a \in M$ and each $\gamma \in \Gamma$ and the left (resp. right) regular Γ -semigroup as a Γ -semigroup in which $a \in M\Gamma a\gamma a$ (resp. $a \in a\gamma a\Gamma M$) for each $a \in M$ and each $\gamma \in \Gamma$ and it is proved that a Γ -semigroup M is left regular (in that new sense) if and only if it is a union of a family of left simple subsemigroups on M . And in [9] we gave some further structure theorems of this type of Γ -semigroups using that new definition and the form of principal filters. But what happens in case of intra-regular and left or right regular *po*- Γ -semigroups? Can we describe the form of the principal filters using some new definitions like in the non-ordered case? The present paper gives the related answer. For more information on Γ (or *po*- Γ)-semigroups cf., for example, the papers in [5–7] of the References, and the papers in which these papers refer. Examples on Γ -semigroups are also given in these papers.

2 On intra-regular ordered *po*- Γ -semigroups

We characterize here the intra-regular *po*- Γ -semigroups in terms of filters, and we prove that a *po*- Γ -semigroup M is intra-regular if and only if the ideals of M are semiprime.

Definition 1. An ordered Γ -semigroup M is called *intra-regular* if

$$x \in (M\Gamma x\gamma x\Gamma M]$$

for every $x \in M$ and every $\gamma \in \Gamma$.

Theorem 2. *An ordered Γ -semigroup M is intra-regular if and only if, for every $x \in M$, we have*

$$N(x) = \{y \in M \mid x \in (M\Gamma y\Gamma M]\}.$$

Proof. \implies . Let $x \in M$ and $T := \{y \in M \mid x \in (M\Gamma y\Gamma M)\}$. Then we have the following:

(1) T is a nonempty subset of M . Indeed: Take an element $\gamma \in \Gamma$ ($\Gamma \neq \emptyset$). Since M is intra-regular, we have

$$x \in (M\Gamma x\gamma x\Gamma M) = \left((M\Gamma x)\gamma x\Gamma M \right] \subseteq \left((M\Gamma M)\Gamma x\Gamma M \right] \subseteq (M\Gamma x\Gamma M],$$

so $x \in T$.

(2) Let $a, b \in T$ and $\gamma \in \Gamma$. Then $a\gamma b \in T$. Indeed: Since $a \in T$, we have $x \in (M\Gamma a\Gamma M]$. Since $b \in T$, we have $x \in (M\Gamma b\Gamma M]$. Since M is intra-regular, $x \in M$ and $\gamma \in \Gamma$, we have $x \in (M\Gamma x\gamma x\Gamma M]$. Then we have

$$\begin{aligned} x \in (M\Gamma x\gamma x\Gamma M] &\subseteq \left(M\Gamma(M\Gamma b\Gamma M]\gamma(M\Gamma a\Gamma M]\Gamma M \right] \\ &= \left(M\Gamma(M\Gamma b\Gamma M)\gamma(M\Gamma a\Gamma M)\Gamma M \right] \\ &= \left((M\Gamma M)\Gamma(b\Gamma M\gamma M\Gamma a)\Gamma(M\Gamma M) \right] \\ &\subseteq \left(M\Gamma(b\Gamma M\gamma M\Gamma a)\Gamma M \right]. \end{aligned}$$

We prove that $b\Gamma M\gamma M\Gamma a \subseteq \left(M\Gamma(a\gamma b)\Gamma M \right]$. Then we have

$$\begin{aligned} x &\in \left(M\Gamma \left(M\Gamma(a\gamma b)\Gamma M \right] \Gamma M \right] = \left(M\Gamma \left(M\Gamma(a\gamma b)\Gamma M \right) \Gamma M \right] \\ &= \left((M\Gamma M)\Gamma(a\gamma b)\Gamma(M\Gamma M) \right] \subseteq \left(M\Gamma(a\gamma b)\Gamma M \right], \end{aligned}$$

so $a\gamma b \in T$. Let now $b\lambda u\gamma v\delta a \in b\Gamma M\gamma M\Gamma a$ for some $u, v \in M$, $\lambda, \delta \in \Gamma$. Since M is intra-regular, for the elements $b\lambda u\gamma v\delta a \in M$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} b\lambda u\gamma v\delta a &\in \left(M\Gamma(b\lambda u\gamma v\delta a)\gamma(b\lambda u\gamma v\delta a)\Gamma M \right] \\ &= \left((M\Gamma b\lambda u\gamma v)\delta(a\gamma b)\lambda(u\gamma v\delta a\Gamma M) \right] \\ &\subseteq \left(M\Gamma(a\gamma b)\Gamma M \right]. \end{aligned}$$

(3) Let $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$. Then $a, b \in T$. Indeed: Since $a\gamma b \in T$, we have $x \in \left(M\Gamma(a\gamma b)\Gamma M \right] \subseteq \left(M\Gamma a\gamma(M\Gamma M) \right] \subseteq (M\Gamma a\Gamma M]$, so $a \in T$. Since $x \in \left(M\Gamma(a\gamma b)\Gamma M \right] \subseteq \left((M\Gamma M)\gamma b\Gamma M \right] \subseteq (M\Gamma b\Gamma M]$, we have $b \in T$.

(4) Let $a \in T$ and $M \ni b \geq a$. Then $b \in T$. Indeed: Since $a \in T$, we have $x \in (M\Gamma a\Gamma M]$. Since $a \leq b$, we have $(M\Gamma a\Gamma M] \subseteq (M\Gamma b\Gamma M]$. Then we have $x \in (M\Gamma b\Gamma M]$, and $b \in T$.

(5) Let F be a filter of M such that $x \in F$. Then $T \subseteq F$. Indeed: Let $a \in T$. Then $x \in (M\Gamma a\Gamma M]$, so $F \ni x \leq u\lambda(a\mu v)$ for some $u, v \in M$, $\lambda, \mu \in \Gamma$. Since F is a filter of M , $x \in F$ and $M \ni u\lambda(a\mu v) \geq x$, we have $u\lambda(a\mu v) \in F$. Since F is a filter of M , $u, a\mu v \in M$, $\lambda \in \Gamma$ and $u\lambda(a\mu v) \in F$, we have $a\mu v \in F$, again since F is a filter of M , $a, v \in M$ and $\mu \in \Gamma$, we have $a \in F$.

\Leftarrow . Let $x \in M$ and $\gamma \in \Gamma$. Then $x \in (M\Gamma x\gamma x\Gamma M]$. Indeed: Since $N(x)$ is a subsemigroup of M , $x \in N(x)$ and $\gamma \in \Gamma$, we have $x\gamma x \in N(x)$. By hypothesis, we get $x \in (M\Gamma(x\gamma x)\Gamma M] = (M\Gamma x\gamma x\Gamma M]$, thus M is intra-regular. \square

Theorem 3. *An ordered Γ -semigroup M is intra-regular if and only if the ideals of M are semiprime.*

Proof. \Rightarrow . Let A be an ideal of M , $x \in M$ and $\gamma \in \Gamma$ such that $x\gamma x \in A$. Since M is intra-regular, we have $x \in (M\Gamma(x\gamma x)\Gamma M] \subseteq ((M\Gamma A)\Gamma M] \subseteq (A\Gamma M] \subseteq (A] = A$, then $x \in A$, and A is semiprime.

\Leftarrow . Let $x \in M$ and $\gamma \in \Gamma$. Then $x \in (M\Gamma x\gamma x\Gamma M]$. In fact: The set $(M\Gamma x\gamma x\Gamma M]$ is an ideal of M . This is because it is a nonempty subset of M , $M\Gamma(M\Gamma x\gamma x\Gamma M] \subseteq (M\Gamma(M\Gamma x\gamma x\Gamma M]) = (M\Gamma(M\Gamma x\gamma x\Gamma M)) \subseteq (M\Gamma x\gamma x\Gamma M]$, $(M\Gamma x\gamma x\Gamma M]\Gamma M \subseteq (M\Gamma x\gamma x\Gamma M]$, and $((M\Gamma x\gamma x\Gamma M]) = (M\Gamma x\gamma x\Gamma M]$. Since $(M\Gamma x\gamma x\Gamma M]$ is semiprime and $(x\gamma x)\gamma(x\gamma x) = x\gamma(x\gamma x)\gamma x \in M\Gamma x\gamma x\Gamma M \subseteq (M\Gamma x\gamma x\Gamma M]$, we have $x\gamma x \in (M\Gamma x\gamma x\Gamma M]$. Again since $(M\Gamma x\gamma x\Gamma M]$ is semiprime, we have $x \in (M\Gamma x\gamma x\Gamma M]$, so M is intra-regular. \square

3 On left regular and left duo po - Γ -semigroups

First we notice that the left (and the right) po - Γ -semigroups are intra-regular. Then we characterize the po - Γ -semigroups which are both left regular and left duo in terms of filters and we prove that a po - Γ -semigroup M is left (resp. right) regular if and only if the left (resp. right) ideals of M are semiprime.

Definition 4. An ordered Γ -semigroup M is called *left regular* (resp. *right*

regular) if

$$x \in (M\Gamma x\gamma x] \text{ (resp. } x \in (x\gamma x\Gamma M])$$

for every $x \in M$ and every $\gamma \in \Gamma$.

Proposition 5. *Let M be an ordered Γ -semigroup. If M is left (resp. right) regular, then M is intra-regular.*

Proof. Let M be left regular, $x \in M$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned} x \in (M\Gamma x\gamma x] &\subseteq \left(M\Gamma(M\Gamma x\gamma x)\gamma x \right] = \left(M\Gamma(M\Gamma x\gamma x)\gamma x \right] \\ &\subseteq \left((M\Gamma M)\Gamma(x\gamma x)\Gamma M \right] \subseteq \left(M\Gamma x\gamma x\Gamma M \right], \end{aligned}$$

thus M is intra-regular. \square

Theorem 6. *An ordered Γ -semigroup M is left regular and left duo if and only if, for every $x \in M$, we have*

$$N(x) = \{y \in M \mid x \in (M\Gamma y]\}.$$

Proof. \implies . Let $x \in M$ and $T := \{y \in M \mid x \in (M\Gamma y]\}$. Since M is left regular, we have $x \in (M\Gamma x\gamma x] \subseteq \left((M\Gamma M)\Gamma x \right] \subseteq (M\Gamma x]$, so $x \in T$, and T is a nonempty subset of M . Let $a, b \in T$ and $\gamma \in \Gamma$. Since $x \in (M\Gamma a]$, $x \in (M\Gamma b]$ and M is left regular, we have

$$\begin{aligned} x \in (M\Gamma x\gamma x] &\subseteq \left(M\Gamma(M\Gamma b)\gamma(M\Gamma a) \right] = \left(M\Gamma(M\Gamma b)\gamma(M\Gamma a) \right] \\ &\subseteq \left(M\Gamma(b\gamma M\Gamma a) \right]. \end{aligned}$$

In addition, $b\gamma M\Gamma a \subseteq (M\Gamma a\gamma b]$. Indeed: Let $b\gamma u\mu a \in b\gamma M\Gamma a$, where $u \in M$ and $\mu \in \Gamma$. Since M is left regular, we have

$$b\gamma u\mu a \in \left(M\Gamma(b\gamma u\mu a)\gamma(b\gamma u\mu a) \right] \subseteq \left(M\Gamma(a\gamma b)\Gamma M \right] = \left((M\Gamma a\gamma b)\Gamma M \right].$$

Since $(M\Gamma a\gamma b]$ is a left ideal, it is a right ideal of M as well, so $(M\Gamma a\gamma b)\Gamma M \subseteq (M\Gamma a\gamma b]$, then $b\gamma u\mu a \in \left((M\Gamma a\gamma b) \right] = (M\Gamma a\gamma b]$. Hence we obtain

$$x \in \left(M\Gamma(M\Gamma a\gamma b) \right] = \left(M\Gamma(M\Gamma a\gamma b) \right] \subseteq \left(M\Gamma(a\gamma b) \right],$$

from which $a\gamma b \in T$.

Let $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$. Since $x \in (M\Gamma a\gamma b] \subseteq (M\Gamma b]$,

we have $b \in T$. Besides, $x \in (M\Gamma a\gamma b] \subseteq ((M\Gamma a]\Gamma M]$. The set $(M\Gamma a]$ as a left ideal, is a right ideal of M as well, so $(M\Gamma a]\Gamma M \subseteq (M\Gamma a]$. Thus we have $x \in ((M\Gamma a]) = (M\Gamma a]$, and $a \in T$.

Let $a \in T$ and $M \ni b \geq a$. Since M is left regular, we have

$$x \in (M\Gamma a\gamma a] \subseteq (M\Gamma b\gamma b] \subseteq ((M\Gamma b]\Gamma M].$$

$(M\Gamma b]$ as a left ideal is a right ideal of M , so $(M\Gamma b]\Gamma M \subseteq (M\Gamma b]$. Hence we have $x \in ((M\Gamma b]) = (M\Gamma b]$, and $b \in T$.

Let F be a filter of M such that $x \in F$ and let $a \in T$. Since $x \in (M\Gamma a]$, we have $F \ni x \leq u\mu a$ for some $u \in M$, $\mu \in \Gamma$. Since F is a filter of M , we have $u\mu a \in F$, and $a \in F$.

\Leftarrow . Let $x \in M$ and $\gamma \in \Gamma$. Since $x \in N(x)$ and $N(x)$ is a subsemigroup of M , we have $x\gamma x \in N(x)$. By hypothesis, we get $x \in (M\Gamma x\gamma x]$, so M is left regular. Let now A be a left ideal of M , $a \in A$, $\gamma \in \Gamma$ and $u \in M$. Since $a\gamma u \in N(a\gamma u)$ and $N(a\gamma u)$ is a filter of M , we have $a \in N(a\gamma u)$. By hypothesis, we have $a\gamma u \in (M\Gamma a] \subseteq (M\Gamma A] \subseteq (A] = A$. Thus A is right ideal of M . \square

The right analogue of Theorem 6 also holds, and we have

Theorem 7. *An ordered Γ -semigroup M is right regular and right duo if and only if, for every $x \in M$, we have*

$$N(x) = \{y \in M \mid x \in (y\Gamma M]\}.$$

Theorem 8. *An ordered Γ -semigroup M is left (resp. right) regular if and only if the left (resp. right) ideals of M are semiprime.*

Proof. \Rightarrow . Let M be left regular, A a left ideal of M , $x \in M$ and $\gamma \in \Gamma$ such that $x\gamma x \in A$. Then we have $x \in (M\Gamma(x\gamma x)] \subseteq (M\Gamma A] \subseteq (A] = A$, so M is semiprime.

\Leftarrow . Suppose the left ideals of M are semiprime and let $x \in M$ and $\gamma \in \Gamma$. Since $(M\Gamma x\gamma x]$ is a left ideal of M and $(x\gamma x)\gamma(x\gamma x) \in (M\Gamma x\gamma x]$, we have $x\gamma x \in (M\Gamma x\gamma x]$, and $x \in (M\Gamma x\gamma x]$, so M is left regular. \square

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